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THE SINGLE-MODE APPROXIMATION IN THE PROBLEM OF THE PROPAGATION OF A PLANE LONGITUDINAL WAVE IN AN ELASTIC MEDIUM WITH A PERIODIC SYSTEM OF RECTANGULAR DEFECTS†

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The approach suggested in [1, 2] is applied to the problem of the propagation of a plan longitudinal wave in an elastic medium containing a periodic system of rectangular defects. Explicit analytical representations for the scattering coefficients as well as a refined low-frequency solution are derived using a uniform approximation of the single-mode type. A comparison of the results with solutions obtained by other methods is given. © 2003 Elsevier Ltd. All rights reserved.

Earlier [1, 2] were used an analytical approach to solve the problem of the scattering of both a transverse wave and a longitudinal wave by a periodic array of rectangular obstacles. The use of analytical methods in the problem of wave propagation in an elastic medium with regularly distributed systems of defects is difficult, and hence such problems are generally analysed numerically by reducing them to infinite systems of algebraic equations [3, 4]. It should also be noted that a periodic array of rectangular defects is equivalent to a screen of finite thickness with a periodic system of apertures.

In [1] the antiplane problem of *SH*-wave propagation was reduced to two independent integral equations on the aperture, and formulae for the scattering parameters were obtained as an explicit function of the frequency using an approximation that holds in the single-mode frequency range. A more difficult case, not previously considered, of the plane problem for the so-called *P*-type wave was analysed in [2]; there the problem was reduced to two 3×3 systems of integral equations along sections coinciding with the sides of the rectangular aperture, the solution of which would have enabled a complete analytical solution for the scattered wave field to be obtained. Only a direct numerical method was applied to this system, which enabled numerical values of all the scattering characteristics to be found [2].

The purpose of this paper is to obtain analytical results for the plane problem with rectangular scatterers in an elastic medium. The suggested single-mode method (developed in [5] for completely different problem), using the previously derived integral equations [2], is extended further here, and enables explicit formulae to be derived for the respective mechanical characteristics, as was done in [1] for the antiplane case. Furthermore, an improved approximation in the law-frequency case is given that is more exact than the trivial solution previously obtained [2, Section 4]. The results of a numerical computation both for the problem of the single-mode approximation and for the low-frequency approximation are given. These results are compared with the exact numerical solution previously derived in [2].

1. FORMULATION OF THE PROBLEM AND REDUCTION TO INTEGRAL EQUATIONS

We will consider an unbounded two-dimensional elastic medium (the case of plane strain) of constant unit density with an infinite system of rectangular defects distributed periodically along the vertical y-axis (Fig. 1). The period of the array is $2a$, the clearance, i.e. the distance between two neighbouring defects, is $2b$, and the length of the horizontal side of the defects is $2l$. A plane longitudinal wave is incident on the array with a displacement potential $\varphi_{inc} = e^{ik_1x}$, propagating along the x axis (see Fig. 1, in which the direction of wave propagation is denoted by an arrow).

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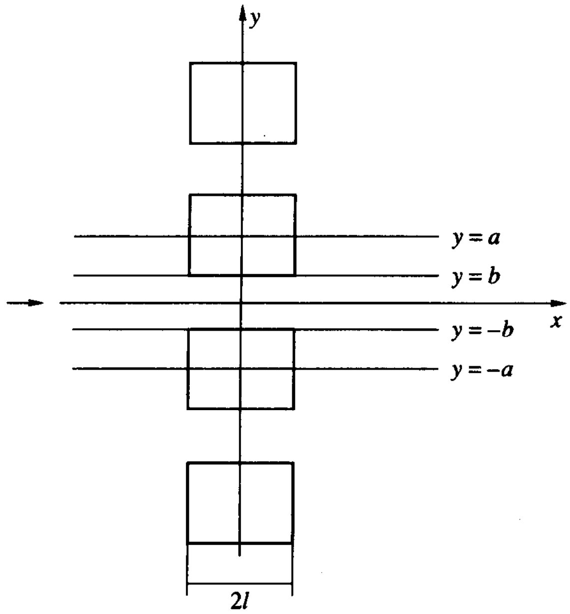

Fig. 1

The common multiplier $e^{-i\omega t}$ (where ω is the angular frequency), implied in all functions occurring in the solution of the problem, can be neglected, assuming that the time-dependence of all the wave characteristics is harmonic. We note also that the natural symmetry and periodicity along the y axis enables us to confine our consideration of the problem to one typical layer $|y| < a$ with a step-like narrowing of the width $|y| < b$ and the length $2l$.

We will write the governing equations of the problem, introducing the field of displacements $\mathbf{u} = (u_x(x, y), u_y(x, y), 0)$, as follows:

the Lamé relations

$$u_x = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad u_y = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{1.1}$$

the Helmholtz equations

$$\Delta\varphi + k_1^2\varphi = 0, \quad \Delta\psi + k_2^2\psi = 0 \tag{1.2}$$

and the equations of state

$$\begin{aligned} \tau_{xy} &= \tau_{yx} = c_2^2\left(2\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \\ \sigma_x &= c_1^2\Delta\varphi - 2c_2^2\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}\right) \\ \sigma_y &= c_1^2\Delta\varphi - 2c_2^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}\right) \end{aligned} \tag{1.3}$$

and add the following natural boundary-value conditions, which are satisfied inside the layer

$$\begin{aligned} &\tau_{xy}(\pm l, y) = \sigma_x(\pm l, y) = 0, \quad b < |y| < a \\ &\tau_{xy}(x, \pm b) = \sigma_y(x, \pm b) = 0, \quad |x| < l; \quad \tau_{xy}(x, \pm a) = u_y(x, \pm a) = 0, \quad |x| > l \end{aligned} \tag{1.4}$$

Thereafter, in all these equations c_1 and c_2 are the longitudinal and transverse velocities of sound in the material considered and k_1 and k_2 are the corresponding wave numbers $k_1 = \omega/c_1 = k_2c_2/c_1$. In this context the single-mode approximation corresponds to the frequency range

$$k_1 < k_2 < \pi/a \tag{1.5}$$

and means that, even though an infinite number of standing waves exists in the vicinity of the array in the medium, only a plane wave with a given wave number, corresponding to the velocity of the longitudinal wave in the given medium, can propagate over long distances. In fact the following asymptotic representation holds for the potentials φ and ψ in the far left and right zones of the layer

$$\varphi_l(x, y) \sim e^{ik_1x} + Re^{-ik_1x}, \quad \psi_l(x, y) \sim 0, \quad x \to -\infty$$

$$\varphi_r(x, y) \sim Te^{ik_1x}, \quad \psi_r(x, y) \sim 0, \quad x \to +\infty$$

for the mode decomposition [2], in view of the fact that all wave numbers of higher order are positive.

The constants R and T obviously define the reflection coefficient and the propagation constant. The purpose of this paper is to construct an explicit dependence of these coefficients on the frequency.

It follows from results previously obtained [2, Section 3] that

$$R = -1 - \frac{1}{2ac_1^2k_1^2}\int_{-b}^{b} g_1^\sigma(y)dy, \quad T = -\frac{1}{2ac_1^2k_1^2}\int_{-b}^{b} g_2^\sigma(y)dy \tag{1.6}$$

where $g_1^\sigma(y)$ and $g_2^\sigma(y)$, $|y| < b$ are certain unknown functions, which are the solution of two 3×3 systems of integral equations.

For convenience we will assume

$$g_1^\pm(y) = g_2^\sigma(y) \pm g_1^\sigma(y), \quad g_2^\pm(y) = g_2^\tau(y) \mp g_1^\tau(y), \quad |y| < b, \quad g_3^\pm(x) = g_u^\pm(x), \quad |x| < l \tag{1.7}$$

where the physical meaning of the functions introduced functions is clear from the following definitions

$$g_1^\sigma(y) \equiv \sigma_x(-l, y), \quad g_2^\sigma(y) \equiv \sigma_x(l, y), \quad g_1^\tau(y) \equiv \tau_{xy}(-l, y), \quad g_2^\tau(y) \equiv \tau_{xy}(l, y)$$

$$g_u^\pm(x) \equiv \frac{1}{2}[u_y(x, b) \pm u_y(-x, b)]$$

Then both systems considered (one denoted by a superscript and the other by a subscript) can be written in the form δ_{ij} is the Kronecker delta)

$$\begin{aligned} &\sum_{j'=1}^{2}\int_{-b}^{b} K_{jj'}^\pm(\eta - y)g_{j'}^\pm(\eta)d\eta + \int_{-l}^{l} K_{j3}^\pm(\xi, y)g_3^\pm(\xi)d\xi = \mp 2ik_1\delta_{1j}, \quad |y| < b, \quad j = 1, 2 \\ &\sum_{j'=1}^{2}\int_{-b}^{b} K_{3j'}^\pm(x, \eta)g_{j'}^\pm(\eta)d\eta + \int_{-l}^{l} K_{33}^\pm(x, \xi)g_3^\pm(\xi)d\xi = 0, \quad |x| < l \end{aligned} \tag{1.8}$$

In this connection 18 of the kernels containing the wave numbers, are

$$K_{11}^\pm(y) = \frac{1}{2abk_1c_1^2}\left[\pm a\begin{Bmatrix}\text{tg}\\ \text{ctg}\end{Bmatrix}(k_1l) + ib\right] - \frac{k_2^2}{ac_2^2}\sum_{n=1}^{\infty}\frac{q_n}{\Delta_n}\cos(a_ny) - \frac{k_2^2}{bc_2^2}\sum_{n=1}^{\infty}\frac{p_n^+}{\Pi_n^\pm}\begin{Bmatrix}S_n\\ C_n\end{Bmatrix}\cos(b_ny)$$

$$K_{12}^\pm(y) = \frac{1}{ac_2^2}\sum_{n=1}^{\infty}\frac{2a_n^2 - k_2^2 - 2q_nr_n}{\Delta_n}a_n\sin(a_ny) + \frac{1}{bc_2^2}\sum_{n=1}^{\infty}\frac{b_nR_n^\pm}{\Pi_n^\pm}\sin(b_ny)$$

$$K_{13}^\pm(x, y) = \frac{2c_2^2 - c_1^2}{bc_1^2\cos(k_1l)}\begin{Bmatrix}\cos(k_1x)\\ 0\end{Bmatrix} \mp \frac{4}{b}\sum_{n=1}^{\infty}\frac{(-1)^n}{\Pi_n^\pm}\begin{Bmatrix} S_nL_n(x) \\ \dfrac{2p_n^{+2} + k_2^2}{k_2^2}b_n^2R_n^-T_n(x)\end{Bmatrix}\cos(b_ny) +$$

$$+ \frac{2}{lk_2^2}\sum_{n=1}^{\infty}(-1)^nl_n\left[\frac{2w_n\text{ch}(w_ny)}{\text{sh}(w_nb)} - \frac{(2l_n^2 - k_2^2)\text{ch}(\upsilon_ny)}{\upsilon_n\text{sh}(\upsilon_nb)}\right]\begin{Bmatrix}0\\ \sin(l_nx)\end{Bmatrix}$$

$$
\begin{aligned}
K_{21}^{\pm}(y) &= K_{12}^{\pm}(y)\\
K_{22}^{\pm}(y) &= \frac{k_2^2}{ac_2^2}\sum_{n=1}^{\infty}\frac{r_n}{\Delta_n}\cos(a_n y) + \frac{k_2^2}{bc_2^2}\sum_{n=1}^{\infty}\frac{p_n^-}{\Pi_n^{\pm}}\begin{Bmatrix} C_n \\ S_n \end{Bmatrix}\cos(b_n y)\\
K_{23}^{\pm}(x,y) &= \frac{2}{lk_2^2}\sum_{n=1}^{\infty}(-1)^n\left[(2l_n^2-k_2^2)\frac{\mathrm{sh}(\upsilon_n y)}{\mathrm{sh}(\upsilon_n b)} - 2l_n^2\frac{\mathrm{sh}(w_n y)}{\mathrm{sh}(w_n b)}\right]\begin{Bmatrix} \cos(l_n x) \\ 0 \end{Bmatrix} -\\
&-\begin{Bmatrix} \dfrac{\sin(k_1 y)}{l\sin(k_1 b)} \\ 0 \end{Bmatrix} \mp \frac{4}{b}\sum_{n=1}^{\infty}\frac{(-1)^n b_n}{\Pi_n^{\pm}}\begin{Bmatrix} \dfrac{R_n^+ L_n(x)}{k_2^2} \\ p_n^-(2p_n^{+2}+k_2^2)S_n T_n(x) \end{Bmatrix}\sin(b_n y)
\end{aligned}
\tag{1.9}
$$

$$
\begin{aligned}
K_{31}^{+}(x,y) &= -\frac{K_{13}^{+}(x,y)}{4c_2^2}\\
K_{31}^{-}(x,y) &= \frac{1}{2bc_2^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{\Pi_n^-}U_n^-(x)\cos(b_n y) - \frac{2c_2^2-c_1^2}{4bc_1^2c_2^2}\frac{\sin(k_1 x)}{\sin(k_1 l)}\\
K_{32}^{\pm}(x,y) &= \frac{1}{bc_2^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{\Pi_n^+}Q_n^{\pm}(x)p_n^- b_n\sin(b_n y)\\
K_{33}^{\pm}(x,\xi) &= \left\{\frac{k_2c_1}{2lc_2}\mathrm{ctg}(k_1 b) - \frac{(2c_2^2-c_1^2)^2 k_1}{bc_1^2c_2^2\sin(2k_1 l)}\cos[k_1(x-\xi)]\right\}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} +\\
&+\frac{1}{lk_2^2}\sum_{n=1}^{\infty}\left[4l_n^2 w_n\mathrm{cth}(w_n b) - \frac{(2l_n^2-k_2^2)^2}{\upsilon_n}\mathrm{cth}(\upsilon_n b)\right]\cos[l_n(x-\xi)] +\\
&+\frac{2}{bk_2^2}\sum_{n=1}^{\infty}\frac{1}{\Pi_n^{\pm}}\begin{Bmatrix} U_n^+(x)L_n(\xi) \\ \dfrac{2b_n^2p_n^-}{k_2^2}(2p_n^{+2}+k_2^2)Q_n^-(x)T_n(\xi) \end{Bmatrix}
\end{aligned}
$$

Here

$$
\begin{aligned}
\begin{Bmatrix} S_n \\ C_n \end{Bmatrix} &= \begin{Bmatrix} \mathrm{sh} \\ \mathrm{ch} \end{Bmatrix}(p_n^+ l)\begin{Bmatrix} \mathrm{sh} \\ \mathrm{ch} \end{Bmatrix}(p_n^- l)\\
R_n^{\pm} &= 2p_n^+p_n^-\mathrm{sh}(p_n^{\pm}l)\mathrm{ch}(p_n^{\mp}l) - (2b_n^2-k_2^2)\mathrm{ch}(p_n^{\pm}l)\mathrm{sh}(p_n^{\mp}l)\\
Q_n^{\pm}(x) &= (2b_n^2-k_2^2)\begin{Bmatrix} \mathrm{ch} \\ \mathrm{sh} \end{Bmatrix}(p_n^+ l)\begin{Bmatrix} \mathrm{ch} \\ \mathrm{sh} \end{Bmatrix}(p_n^- x) - (2b_n^2+k_2^2)\begin{Bmatrix} \mathrm{ch} \\ \mathrm{sh} \end{Bmatrix}(p_n^- l)\begin{Bmatrix} \mathrm{ch} \\ \mathrm{sh} \end{Bmatrix}(p_n^+ x)\\
U_n^{\pm}(x) &= 4b_n^2p_n^+p_n^-\begin{Bmatrix} \mathrm{sh} \\ \mathrm{ch} \end{Bmatrix}(p_n^+ l)\begin{Bmatrix} \mathrm{ch} \\ \mathrm{sh} \end{Bmatrix}(p_n^- x) - (2b_n^2-k_2^2)(2p_n^{+2}+k_2^2)\begin{Bmatrix} \mathrm{sh} \\ \mathrm{ch} \end{Bmatrix}(p_n^- l)\begin{Bmatrix} \mathrm{ch} \\ \mathrm{sh} \end{Bmatrix}(p_n^+ x)\\
L_n(x) &= \frac{2b_n^2p_n^+p_n^-}{\mathrm{sh}(p_n^- l)}\mathrm{ch}(p_n^- x) - \frac{(2b_n^2-k_2^2)(2p_n^{+2}+k_2^2)}{2\,\mathrm{sh}(p_n^+ l)}\mathrm{ch}(p_n^+ x)\\
T_n(x) &= \frac{\mathrm{sh}(p_n^- x)}{\mathrm{sh}(p_n^- l)} - \frac{\mathrm{sh}(p_n^+ x)}{\mathrm{sh}(p_n^+ l)}
\end{aligned}
\tag{1.10}
$$

$$a_n = \frac{\pi n}{a},\quad b_n = \frac{\pi n}{b},\quad l_n = \frac{\pi n}{l},\quad q_n = \sqrt{a_n^2-k_1^2},\quad r_n = \sqrt{a_n^2-k_2^2},\quad p_n^+ = \sqrt{b_n^2-k_1^2}$$

$$p_n^- = \sqrt{b_n^2-k_2^2},\quad v_n = \sqrt{l_n^2-k_1^2},\quad w_n = \sqrt{l_n^2-k_2^2},\quad \Delta_n = (2a_n^2-k_2^2)^2-4a_n^2q_nr_n$$

$$\Pi_n^{\pm} = (2b_n^2-k_2^2)^2\operatorname{ch}(p_n^{\pm}l)\operatorname{sh}(p_n^{\mp}l)-4b_n^2p_n^+p_n^-\operatorname{sh}(p_n^{\pm}l)\operatorname{ch}(p_n^{\mp}l);\quad n = 1,2,\ldots$$

2. THE SINGLE-MODE APPROXIMATION AND EXPLICIT SOLUTIONS

We will apply an approximation to the kernels (1.9), in which they will be represented in a form not containing the wave numbers. In this connection we will assume that: (1) $l \leqslant a$, i.e. the length of the rectangles does not exceed the lattice period, (2) the following approximate formulae hold

$$\begin{gathered} q_n \approx r_n \approx a_n,\quad p_n^{\pm} \approx b_n,\quad v_n \approx w_n \approx l_n,\quad q_nr_n \approx a_n^2-(k_1^2+k_2^2)/2 \\ p_n^+p_n^- \approx b_n^2-(k_1^2+k_2^2)/2,\quad v_nw_n \approx l_n^2-(k_1^2+k_2^2)/2,\quad n = 1,2,\ldots \end{gathered} \tag{2.1}$$

which are identical with the main terms of the asymptotic form of the corresponding expressions for large n in the domain of frequencies, defined by Eq. (1.5). The equations

$$\Delta_n \approx 2a_n^2(k_1^2-k_2^2),\quad \Pi_n^{\pm} \approx 2b_n^2(k_1^2-k_2^2)\operatorname{sh}(b_nl)\operatorname{ch}(b_nl),\quad n = 1,2,\ldots \tag{2.2}$$

and also the analogous asymptotic approximations for all the other expressions appearing in the kernels, follow from Eq. (2.1). As an example we will only present two of them

$$\begin{gathered} 4l_n^2w_n\operatorname{cth}(w_nb)-\frac{(2l_n^2-k_2^2)^2}{v_n}\operatorname{cth}(v_nb) \approx 2(k_2^2-k_1^2)l_n\operatorname{cth}(l_nb) \\ U_n^{\pm}(x) \approx 2b_n^2(k_1^2-k_2^2)\begin{Bmatrix}\operatorname{ch}\\ \operatorname{sh}\end{Bmatrix}(b_nl)\begin{Bmatrix}\operatorname{sh}\\ \operatorname{ch}\end{Bmatrix}(b_nx) \end{gathered} \tag{2.3}$$

Note that, in the standard single-mode approximation, Eqs (2.1) hold only for $n \geqslant 2$, but must be retained in the exact form for the case $n = 1$. However, approximation (2.1) is more exact than the usual low-frequency approximation and is not reduced to the latter, since, for instance, we have $r_1/a_1 = \sqrt{1-(ak_2/\pi)^2}$ for $ak_2 = \pi/4$. Thus, the approximation considered in this paper should be called "almost single-mode", but for simplicity we will retain the traditional term "single-mode approximation".

Both 3×3 systems can be reduced to a form in which the kernels are free from the frequency parameters k_1 and k_2, taking into account the approximations assumed. The following tabulated series must be used for this

$$\sum_{n=1}^{\infty}\frac{\cos(a_ny)}{n} = -\ln\left|2\sin\frac{\pi y}{2a}\right|,\quad \sum_{n=1}^{\infty}\frac{\sin(a_ny)-\sin(b_ny)}{n} = \frac{\pi(a-b)}{2ab}y,\quad |y|<2a \tag{2.4}$$

As a result the systems considered in the given single-mode approximation can be reduced to the form (in all the equations $|y| < b$, $|x| < l$)

$$\frac{c_1^2}{2\pi c_2^2(c_2^2-c_1^2)}\int_{-b}^{b} g_1^{\pm}(\eta)\left[\ln\left|2\sin\frac{\pi(\eta-y)}{2a}\right|-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{\cos[b_n(\eta-y)]}{n}\begin{Bmatrix}\operatorname{th}\\ \operatorname{cth}\end{Bmatrix}(b_nl)\right]d\eta+\frac{a-b}{4ab(c_2^2-c_1^2)}\int_{-b}^{b} g_2^{\pm}(\eta)(\eta-y)d\eta-$$

$$-\begin{Bmatrix}\dfrac{2}{b}\\ \dfrac{2c_2^2}{lc_1^2}\end{Bmatrix}\int_{-l}^{l} g_3^{\pm}\sum_{n=1}^{\infty}(-1)^n\begin{Bmatrix}\dfrac{\operatorname{ch}(b_n\xi)\cos(b_ny)}{\operatorname{ch}(b_nl)}\\ \dfrac{\operatorname{ch}(l_ny)\sin(l_n\xi)}{\operatorname{sh}(l_nb)}\end{Bmatrix}d\xi = \mp 2ik_1-\frac{1}{2abk_1c_1^2}\left[ib\pm\begin{Bmatrix}\operatorname{tg}\\ \operatorname{ctg}\end{Bmatrix}(k_1l)\right]G_\sigma^{\pm}+$$

$$+\frac{(2c_2^2-c_1^2)}{bc_1^2\text{ctg}(k_1l)}\left[\frac{k_1}{2l}\begin{Bmatrix}G_{u2}^+\\0\end{Bmatrix}-\chi\begin{Bmatrix}G_u^+\\0\end{Bmatrix}\right] \tag{2.5}$$

$$\frac{(a-b)}{4ab(c_2^2-c_1^2)}\int_{-b}^{b}g_1^{\pm}(\eta)(\eta-y)d\eta-\frac{c_1^2}{2\pi c_2^2(c_2^2-c_1^2)}\int_{-b}^{b}g_2^{\pm}(\eta)\left[\ln\left|2\sin\frac{\pi(\eta-y)}{2a}\right|-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{\cos[b_n(\eta-y)]}{n}\begin{Bmatrix}\text{cth}\\\text{th}\end{Bmatrix}(b_nl)\right]d\eta+\begin{Bmatrix}2\\0\end{Bmatrix}\int_{-l}^{l}g_3^{\pm}(\xi)\sum_{n=1}^{\infty}(-1)^n\times$$

$$\times\left[\frac{c_2^2\text{ch}(b_n\xi)}{bc_1^2\text{sh}(b_nl)}\sin(b_ny)-\frac{\text{sh}(l_ny)}{l\,\text{sh}(l_nb)}\cos(l_n\xi)\right]d\xi=\left(\frac{y}{lb}+\frac{2k_1^2}{\pi lp_1^2}\sin\frac{\pi y}{b}\right)\begin{Bmatrix}G_u^+\\0\end{Bmatrix} \tag{2.6}$$

$$\int_{-b}^{b}g_1^{\pm}(\eta)\sum_{n=1}^{\infty}(-1)^n\cos(b_n\eta)\begin{Bmatrix}\dfrac{\text{ch}(b_nx)}{\text{ch}(b_nl)}\\[2mm]\dfrac{\text{sh}(b_nx)}{\text{sh}(b_nl)}\end{Bmatrix}d\eta+2\int_{-b}^{b}g_2^{\pm}(\eta)\sum_{n=1}^{\infty}(-1)^n\sin(b_n\eta)\begin{Bmatrix}\dfrac{\text{ch}(b_nx)}{\text{sh}(b_nl)}\\[2mm]\dfrac{\text{sh}(b_nx)}{\text{ch}(b_nl)}\end{Bmatrix}d\eta+$$

$$+\frac{4c_2^2(c_1^2-c_2^2)}{lc_1^2}\int_{-l}^{l}g_3^{\pm}(\xi)\left[\sum_{n=1}^{\infty}bl_n\text{cth}(l_nb)\cos[l_n(x-\xi)]-\begin{Bmatrix}1\\0\end{Bmatrix}\sum_{n=1}^{\infty}\frac{lb_n\text{ch}(b_nx)\text{ch}(b_n\xi)}{\text{sh}(b_nl)\text{ch}(b_nl)}\right]d\xi=$$

$$=\frac{2c_2^2-c_1^2}{2c_1^2}\left[\text{tg}(k_1l)\left(\chi-\frac{k_1}{2l}x^2\right)\begin{Bmatrix}G_\sigma^+\\0\end{Bmatrix}+\left(\frac{x}{l}+\frac{2k_1^2}{\pi v_1^2}\sin\frac{\pi x}{l}\right)\begin{Bmatrix}0\\G_\sigma^-\end{Bmatrix}\right]+$$

$$+\left[\frac{(2c_2^2-c_1^2)^2k_1}{c_1^2\cos(k_1l)}\left(\chi-\frac{k_1}{2l}x^2\right)-\frac{bc_1c_2k_2\text{ctg}(k_1b)}{l}\right]\begin{Bmatrix}G_u^+\\0\end{Bmatrix}-\frac{(2c_2^2-c_1^2)^2k_1^2}{2lc_1^2\cos(k_1l)}\begin{Bmatrix}G_{u2}^+\\0\end{Bmatrix} \tag{2.7}$$

where

$$\chi=\frac{6+k_1^2l^2}{6k_1l},\quad G_\sigma^{\pm}=\int_{-b}^{b}g_1^{\pm}(\eta)d\eta=\int_{-b}^{b}[g_2^{\sigma}(\eta)\pm g_1^{\sigma}(\eta)]d\eta$$

$$G_u^+=\int_{-l}^{l}g_3^+(\xi)d\xi=\int_{-l}^{l}g_u^+(\xi)d\xi,\quad G_{u2}^+=\int_{-l}^{l}g_u^+(\xi)\xi^2d\xi \tag{2.8}$$

($G_\sigma^{\pm}$, G_u^+ and G_{u2}^+ are certain unknown constants). It also should be noted that all kernels described are real and do not contain the frequency parameters k_1 and k_2.

When deriving Eqs (2.5)–(2.7) the terms in which the wave number and the space coordinate are present at the same time are the main difficulty, since an explicit separation of the frequency parameters is impossible in these terms. These functions were decomposed into Fourier series in which the standard single-mode approximation $|x| < l$ was then used to overcome this difficulty

$$\sin(k_1x)=-\frac{2\pi}{l^2}\sin(k_1l)\sum_{n=1}^{\infty}\frac{(-1)^nn}{v_n^2}\sin(l_nx)\approx$$

$$\approx-\frac{2\pi}{l^2}\sin(k_1l)\left[-\frac{1}{v_1^2}\sin(l_1x)+\sum_{n=2}^{\infty}\frac{(-1)^nn}{l_n^2}\sin(l_nx)\right]=\sin(k_1l)\left(\frac{x}{l}+\frac{2k_1^2}{\pi v_1^2}\sin\frac{\pi x}{l}\right) \tag{2.9}$$

and similarly $|t| < 2l$

$$\cos(k_1 t) = \frac{\sin(k_1 l)}{k_1 l} - \frac{2k_1 l\sin(k_1 l)}{l^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{v_n^2}\cos(l_n t) \approx$$
$$\approx \sin(k_1 l)\left(\frac{1}{k_1 l} - \frac{2k_1 l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(l_n t)\right) = \sin(k_1 l)\left(\chi - \frac{k_1}{2l}t^2\right) \tag{2.10}$$

Here the following summation formulae were used

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(l_n x) = -\frac{\pi x}{2l}, \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(l_n t) = \frac{\pi^2 t^2}{4l^2} - \frac{\pi^2}{12}$$

We will now write Eqs (2.5)–(2.7) in symbolic form (everywhere henceforth Σ' denotes summation from $j' = 1$ to $j' = 3$)

$$\sum{}' K^{\pm}_{jj'} g^{\pm}_{j'} = f^{\pm}_{j}, \quad j = 1, 2, 3 \tag{2.11}$$

and introduce auxiliary real functions h^-_{ij} ($i = 1, 2, 3$) as a solution of the following systems, which do not include the oscillation frequency

$$\sum{}' K^-_{jj'} h^-_{1j'} = \delta_{1j}, \quad \sum{}' K^-_{jj'} h^-_{2j'} = \delta_{3j}\frac{x}{l}, \quad \sum{}' K^-_{jj'} h^-_{3j'} = \delta_{3j}\sin\frac{\pi x}{l} \tag{2.12}$$

Then, by virtue of the linearity, we obtain

$$g^-_j = \left[2ik_1 + \frac{a\,\mathrm{ctg}(k_1 l) - ib}{2abk_1 c_1^2}G^-_\sigma\right]h^-_{1j} + \frac{2c_2^2 - c_1^2}{2c_1^2}G^-_\sigma\left(h^-_{2j} + \frac{2k_1^2}{\pi v_1^2}h^-_{3j}\right) \tag{2.13}$$

It is obvious that the constant G^-_σ (2.8) can be determined from Eq. (2.13) by integrating over the interval $|\eta| \leqslant b$, which leads to the expression

$$G^-_\sigma = 2ik_1 H^-_{11}\left[1 - \frac{a\,\mathrm{ctg}(k_1 l) - ib}{2abk_1 c_1^2}H^-_{11} - \frac{2c_2^2 - c_1^2}{4c_1^2}\left(H^-_{21} + \frac{2k_1^2}{\pi v_1^2}H^-_{31}\right)\right]^{-1} \tag{2.14}$$

in which the constants

$$H^-_{i1} = \int_{-b}^{b} h^-_{i1}(\eta)d\eta, \quad i = 1, 2, 3 \tag{2.15}$$

do not contain the wave number.

The (3×3) system corresponding to the superscript + is analysed similarly.

We will introduce the following parameters, which depend on the frequency

$$A = -\frac{a\,\mathrm{tg}(k_1 l) + ib}{2abk_1 c_1^2}, \quad B = -\frac{2c_2^2 - c_1^2}{bc_1^2\mathrm{ctg}(k_1 l)}\chi, \quad C = \frac{2c_2^2 - c_1^2}{bc_1^2\mathrm{ctg}(k_1 l)}\frac{k_1}{2l}$$
$$D = \frac{2k_1^2}{\pi l p_1^2}, \quad E = \frac{(2c_2^2 - c_1^2)^2 k_1}{2bc_1^2 c_2^2\cos(k_1 l)}\chi - \frac{c_1 k_2\mathrm{ctg}(k_1 b)}{2lc_2}, \quad F = -\frac{(2c_2^2 - c_1^2)^2 k_1}{2bc_1^2 c_2^2\cos(k_1 l)}\frac{k_1}{2l} \tag{2.16}$$
$$G = \frac{(2c_2^2 - c_1^2)\mathrm{tg}(k_1 l)}{4bc_1^2 c_2^2}\chi, \quad H = -\frac{(2c_2^2 - c_1^2)\mathrm{tg}(k_1 l)k_1}{4bc_1^2 c_2^2}\frac{k_1}{2l}$$

Then the right-hand sides f_j^+ in system (2.11) take the form

$$
\begin{aligned}
f_1^+ &= AG_\sigma^+ + BG_u^+ + CG_{u2}^+ - 2ik_1 \\
f_2^+(y) &= \left(\frac{y}{bl} + D\sin\frac{\pi y}{b}\right)G_u^+ \\
f_3^+(x) &= GG_\sigma^+ + EG_u^+ + FG_{u2}^+ + (HG_\sigma^+ + FG_u^+)x^2
\end{aligned} \tag{2.17}
$$

As above we will introduce auxiliary real functions h_{ij}^+ ($j = 1, 2, 3, 4, 5$) as a solution of the following systems, not containing the frequency parameter

$$
\begin{aligned}
&\sum{}' K_{jj'}^+ h_{1j'}^+ = \delta_{1j}, \quad \sum{}' K_{jj'}^+ h_{2j'}^+ = \delta_{2j}\frac{y}{b}, \quad \sum{}' K_{jj'}^+ h_{3j'}^+ = \delta_{2j}\sin\frac{\pi y}{b} \\
&\sum{}' K_{jj'}^+ h_{4j'}^+ = \delta_{3j}, \quad \sum{}' K_{jj'}^+ h_{5j'}^+ = \delta_{3j}x^2
\end{aligned} \tag{2.18}
$$

Then by virtue of the linearity of the problem we have

$$
\begin{aligned}
g_j^+ &= (AG_\sigma^+ + BG_u^+ + CG_{u2}^+ - 2ik_1)h_{1j}^+ + G_u^+\left(\frac{1}{l}h_{2j}^+ + Dh_{3j}^+\right) + \\
&+ (GC_\sigma^+ + EG_u^+ + FG_{u2}^+)h_{4j}^+ + (HG_\sigma^+ + FG_u^+)h_{5j}^+, \quad j = 1, 2, 3
\end{aligned} \tag{2.19}
$$

By integrating Eq. (2.19) with respect to $|\eta| < b$ for $j = 1$ and then with respect to $|\xi| < l$ for $j = 3$ and finally multiplying it by ξ^2, integrating once more with respect to $j = 3$ for $|\xi| < l$, we obtain that the given solution leads to an algebraic system in the unknown variables G_σ^+, G_u^+ and G_{u2}^+ containing, apart from constants $A, B, \ldots, H$, the constants

$$
H_{k1}^+ = \int_{-b}^{b} h_{k1}^+(\eta)d\eta, \quad H_{k3}^+ = \int_{-l}^{l} h_{k3}^+(\xi)d\xi, \quad H_{k3}^{+2} = \int_{-l}^{l} h_{k3}^+(\xi)\xi^2 d\xi, \quad k = 1, \ldots, 5 \tag{2.20}
$$

which do not depend on the wave numbers.

We obtain the frequency-dependence in explicit form by solving this system for the unknown variable G_σ^+. Note that the frequency only appears in the constant $A, B, \ldots, H$ (2.16). Since

$$
\int_{-b}^{b} g_1^\sigma(\eta)d\eta = \frac{1}{2}(G_\sigma^+ - G_\sigma^-), \quad \int_{-b}^{b} g_2^\sigma(\eta)d\eta = \frac{1}{2}(G_\sigma^+ + G_\sigma^-) \tag{2.21}
$$

the required explicit solution for the scattering coefficients can be obtained in the final form, taking into account Eqs (1.6) and (2.14).

3. THE LOW-FREQUENCY APPROXIMATION

To obtain asymptotic results for limitingly low frequencies, we will first consider Eq. (2.14) for G_σ^- as $k_1 \to 0$: the main term of the asymptotic form (both for the real part and for the imaginary part) equals $[a \operatorname{ctg}(k_1 l) - ib]/(2abk_1c_1^2)$; consequently, the required low-frequency approximation is

$$
G_\sigma^- = \frac{4abc_1^2k_1^2}{b + ia\operatorname{ctg}(k_1 l)} \tag{3.1}
$$

As regard the algebraic system for determining the constant G_σ^+, it follows from Eq. (2.1) that the parameters C, D, F and H tend to zero as k_1^2 as $k_1 \to 0$. Consequently, the constant G_σ^+ can be found from the 2×2 algebraic system, by solving which we obtain

$$
\begin{aligned}
G_\sigma^+ &= 2ik_1[H_{11}^+(\Phi_3^+ - 1) - H_{13}^+\Phi_1^+][(\Psi_1^+ - 1)(\Phi_3^+ - 1) - \Psi_3^+\Phi_1^+]^{-1} \\
\Phi_k^+ &= BH_{1k}^+ + \frac{1}{l}H_{2k}^+ + EH_{4k}^+, \quad \Psi_k^+ = AH_{1k}^+ + GH_{4k}^+, \quad k = 1, 2, 3
\end{aligned} \tag{3.2}
$$

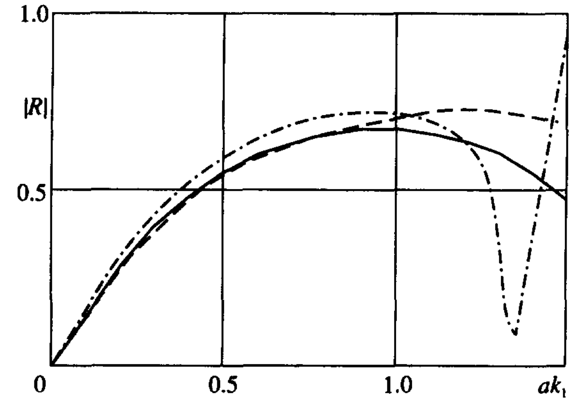

Fig. 2

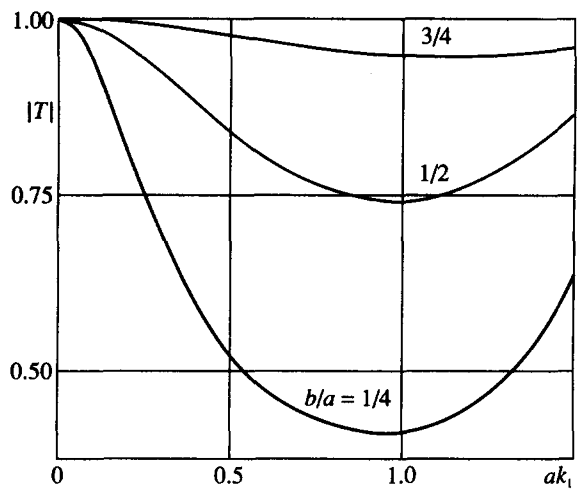

Fig. 3

We obtain the correct low-frequency approximation for the scattering coefficients by substituting the quantities G_σ^+ obtained above into Eq. (2.21) and using expression (1.6).

Compared to the results obtained previously [4, Eqs (4.20) and (4.21)] only the constant G_σ^- (3.1) in the low-frequency limit is expressed in elementary form. The determination of the constant G_σ^+ requires solutions of certain integral equations not containing the frequency parameter.

4. NUMERICAL RESULTS

A direct numerical method was applied to the system of integral equations (2.12) and (2.18), as a result of which values of all the constants H_{j1}^- ($j = 1, 2, 3$) and $H_{j1}^+, H_{j3}^+, H_{j3}^{+2}$ ($j = 1, \ldots, 5$), appearing in the relation for determining constants G_σ^- and G_σ^+, were obtained. Aluminium was taken as the elastic material, for which $c_1 = 6200$ M/s, $c_2 = 3080$ M/s (so that $ak_1 = (c_2/c_1)ak_2 \leqslant ak_2/2 \leqslant 3/2$).

The behaviour of the reflection coefficient as a function of frequency in the single-mode regime (1.5) for the case of square obstacles $b/a = l/a = 1/2$ is shown in Fig. 2. The solid curve corresponds to the explicit formula obtained above in the single-mode approximation, namely: the quantity $|R|$ as a function of ak_1 is taken according to relations (1.6) and (2.21). The dashed curve shows the low-frequency approximation. The dash–dot curve corresponding to the exact solution derived by the direct numerical method [2, line *1* in Fig. 2] is also given. It is obvious that the low-frequency approximation for $|R|$, derived in the present paper, is much more exact than the one previously obtained [2, line *4* in Fig. 2] and it also has a true asymptotic behaviour at limitingly low frequencies.

In Fig. 3 we show the single-mode approximation for the propagation constant as a function of the frequency for three particular values of the relative aperture b/a; for all the curves $l/a = 1/2$. Apart from the obvious fact that an array with a smaller aperture produces less transmission, we note that a comparison of the curve for $b/a = 1/2$ in Fig. 3 with the solid curve in Fig. 2 shows extremely strict satisfaction of the well-known energy balance relation $|R|^2 + |T|^2 = 1$ in the single-mode case [2, 6].

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